Stochastic resonance: Noise-enhanced phase coherence

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We study stochastic resonance in periodically driven stochastic bistable systems in terms of phase synchronization. By introduction of an instantaneous phase for the output we show explicitly the effect of phase locking between the input and output. The stochastic dynamics of the phase difference between input and output appears to be similar to that of synchronized classical self-sustained oscillators. The degree of phase coherence is estimated by employing the effective diffusion constant for the phase difference. This coherence becomes maximal for optimal noise intensities. However, phase synchronization effects can only be observed for sufficiently large magnitude of the periodic inputting signal. [S1063-651X(98)11812-3]

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I. INTRODUCTION

The phenomenon of stochastic resonance (SR) [1] has been extensively studied over the last two decades [2,3]. SR occurs in a wide class of nonlinear systems driven simultaneously by noise and a signal. The necessary property which a nonlinear system should possess to be able to demonstrate SR is the existence of a noise-controlled time scale.

The traditional description of SR defines this effect as amplification of a weak signal applied to the input of the system by tuning the noise intensity. SR manifests itself in the existence of a bell-shaped maximum in the dependence of the spectral power amplification (SPA) [4] or of the signal-to-noise ratio (SNR) [5] versus noise intensity. For extremely weak signals SR is correctly described by linear response theory [6]. In this case a stochastic resonator might be thought of as an equivalent filter with a noise-tuned transfer function determined by the linear susceptibility of the system. In order to calculate the response of the system we have to know its statistical properties in an unperturbed stationary state (i.e., in the absence of signal). From this point of view the structure of weak signals is immaterial: the signal can be harmonic, quasiperiodic [7], or even aperiodic broadband noisy [8,9].

An alternative description of SR, based on the statistics of residence times, has been proposed in [10,11] and characterizes SR as a kind of synchronization of the switching events by external periodic signal. Based on an accurate systematic theory this approach has been reconsidered recently by Choi *et al.* [12]. In the absence of the periodic excitation the residence time distribution possesses an exponential shape. However, when the periodic signal is switched on and its amplitude is sufficiently strong, the residence time distribution becomes structured and contains series of peaks centered at the odd multiples of the half period of the signal. At an optimal noise level the peak at the half driving period becomes dominant and its height with subtracting exponential background [12] passes through a maximum by varying the noise intensity.

For vanishingly small driving amplitudes the residence time distribution is not structured at all [12]. That is, the residence time distribution indeed refers to nonlinear effects with respect to the signal amplitude. However, it provides no information about the instantaneous matching of outputting switching events and of the input signal. For how long are noise-induced switchings between metastable states in synchrony with periodic input? Is it possible to observe frequency locking effects in finite regions in the parameter space of the system as it is in classical self-sustained oscillators? It is important that such a formulation of the problem is just the same as in classical theory of oscillations, where synchronization is originally understood as instantaneous matching of the input/output phases. The positive answer to the second question has recently already been given in [13] and in [14] where the phenomena of a mean switching frequency locking have been reported for periodically driven and coupled stochastic bistable systems, respectively. Arnold tongues of synchronized states were also observed in investigations of periodically driven noisy excitable systems [15].

The goal of the present study is to bridge between the classical notion of synchronization as instantaneous phase locking [16] and synchronizationlike effects occurring in SR systems. For this purpose we first go back to the classical definition of synchronization in Sec. II. In Sec. III we discuss various definitions of an instantaneous phase for periodically driven stochastic bistable systems. The effects of phase and frequency locking are discussed in Sec. IV and compared with other descriptions of SR in Sec. V.

II. SYNCHRONIZATION IN SELF-SUSTAINED OSCILLATORS

The fundamental phenomenon of synchronization [16] occurs in coupled or periodically forced nonlinear self-sustained oscillators. In the absence of the periodic force or if uncoupled the system should possess a stable limit cycle in the phase space which reflects stable oscillations occurring in the system. The properties of these oscillations, i.e., their natural frequency and their amplitude, are determined by their internal dynamics, only, and do not depend (in some reasonable ranges) on initial conditions.

Synchronization can be defined as the locking between the instantaneous phases $\Phi(t)$ of a state variable of the oscillator and the phase $\Psi(t)=\Omega_0 t$ of the external periodic force, i.e., if $|n\Phi(t)-m\Psi(t)|<$const. A weaker condition requires frequency locking $\Omega=\dot{\Phi}=(m/n)\Omega_0$. In both cases m,n are integer numbers. In the parameter space of the system these requirements are fulfilled in finite regions called Arnold tongues. Thereby, the onset of synchronization corresponds to a birth of resonant limit cycles lying on a two dimensional torus in the phase space of the system. Further we will restrict to the simplest case of 1:1 mode locking $(m=n=1)$.

To the best of our knowledge the topic of the influence of noise on synchronizing self-sustained oscillators was first raised by Ritov [17]. The effect of Gaussian white noise was studied in detail by Stratonovich [18] and generalized to colored noise [19]. Inclusion of noise in periodically forced self-sustained oscillators led to amplitude and phase fluctuations [18]. As a result, the phase difference $\phi(t)=\Phi(t)-\Psi(t)$ also fluctuates. If the external periodic force is contaminated by additive Gaussian noise and under the assumption of a constant amplitude its slow dynamics can be described by the stochastic differential equation (SDE)

$$\dot{\phi}=\Delta-\epsilon G(\phi)+\xi(t). \tag{1}$$

Here $\Delta=\Omega-\Omega_0$ is the frequency mismatch, $G(\phi)$ is a 2π periodic function, ϵ is the parameter of nonlinearity, and $\xi(t)$ is noise. In the case of a van der Pol oscillator $G(\phi)\equiv\sin\phi$ and the phase difference performs overdamped Brownian motion in the tilted periodic potential $U(\phi)=-\Delta\phi-\epsilon\cos\phi$ [18]. If $\Delta<\epsilon$ and the noise strength is small the phase difference fluctuates for a long time inside a well of the potential $U(\phi)$ (that means phase locking). It rarely makes jumps from one potential well to another one (i.e., displays phase slips).

The definition of synchronization in the presence of noise appears to be "blurred." For noisy systems one has to use the notion of an *effective synchronization* [20]. It can be defined via imposing restrictions to (i) phase fluctuations, (ii) frequency fluctuations, and (iii) output signal-to-noise ratio [20] whereby the conditions in this sequence of restrictions are subsequently lowered. Note that SR measures based on the residence time distribution refer to the second type of definition.

Further on, our investigations use the strongest definition of effective synchronization based on the statistics of phase fluctuations. We assert that a noisy system is effectively synchronized to an external periodic force if T_{mean} is larger than some given value, where T_{mean} is the mean time in the course of which the instantaneous phase of the system is locked. We will require that this mean time between two phase slips should be greater than large multiples of the driving period $T_{\max}=n2\pi/\Omega_0$, $n\gg1$.

A quantity related to this definition and which will be used later on as the measure of phase coherence represents D_{eff}, is defined as

$$D_{\mathrm{eff}}=\frac{1}{2}\frac{d}{dt}[\langle\phi^2(t)\rangle-\langle\phi(t)\rangle^2]. \tag{2}$$

This is the effective diffusion constant describing the spreading of an initial distribution of the phase difference along the potential profile. It can be shown that D_{eff} is proportional to the mean escape rate r from a well of the potential $U(\phi)$, i.e., $D_{\mathrm{eff}}=4\pi^2 r$ [18]. Hence, D_{eff} is proportional to the inverse mean time interval of a locked phase difference. As a result, effective synchronization of noisy oscillators is achieved, if

$$D_{\mathrm{eff}}\leqslant 2\pi\frac{\Omega_0}{n}. \tag{3}$$

This condition determines regions of effective synchronization in the parameter space of the system.

The influence of additive noise on synchronized self-sustained oscillators is well known [18,19]: the effective diffusion constant grows with the increase of the noise strength, e.g., phase slips appear more frequently. In other words, noise acts against synchronization leading to the loss of phase coherence and shrinks Arnold tongues [21,22].

In the next sections we show that SR systems display just an opposite behavior: With the increase of the noise intensity the degree of phase coherence first grows and only for sufficiently large noise does the system become asynchronous. Hence, SR systems become effectively synchronized in a finite region of optimally selected noise intensities.

III. INSTANTANEOUS PHASE FOR PERIODICALLY DRIVEN STOCHASTIC BISTABLE SYSTEMS

An overdamped stochastic bistable oscillator, the most popular example of a SR system, obviously does not have a deterministic frequency. Instead, it possesses a noise-controlled time scale represented by the Kramers time or mean escape time from a potential well and has essentially relaxation features. In the frequency domain this time scale determines the mean switching frequency (MSF) of the system. A periodic signal subjected to the input of a stochastic resonator represents therefore a single external "clock" [3] which is amenable to synchronize the switchings between the metastable states of the system.

Further, we numerically treat this overdamped Kramers oscillator driven by an external periodic force with frequency Ω. In canonical units it is governed by the SDE [23]

$$\dot{x}=x-x^3+\sqrt{2D}\xi(t)+A\cos(\Omega_0 t+\psi_0), \tag{4}$$

where $\xi(t)$ is white Gaussian noise D scales the intensity of this noise and ψ_0 is the initial phase of the signal. We set $\psi_0=0$ for convenience. The amplitude A of the periodic force is always sufficiently small: the signal cannot switch the system from one state to the other one in the absence of noise. For the low-frequency modulation this requires that

$$A<A_0=\frac{2}{3\sqrt{3}}\propto 0.3849\ldots. \tag{5}$$

The case of suprathreshold values $(A>A_0)$ has been studied recently in [24] in connection with the phenomenon of resonant trapping.

In order to study synchronization in the above described classical sense we need to introduce an instantaneous phase of the system. It is well known that for aperiodic signals the definition of the phase becomes ambiguous.

A formal but general definition of an instantaneous phase is based on the concept of an analytic signal introduced by Gabor [25]. It is widely used in the theory of nonlinear oscillations and waves [26,27] as well as in communication theory [28]. Recently, the definition of an instantaneous phase by this concept has been applied to the study of phase synchronization of chaotic systems [29]. The analytic signal $w(t)$ is a complex function of time defined as

$$w(t)=x(t)+iy(t)=a(t)e^{i\Phi(t)}. \tag{6}$$

Here $y(t)$ is the Hilbert transform (HT) of the original process $x(t)$,

$$y(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau. \tag{7}$$

In the latter expression the integral is taken in the sense of a Cauchy principal value. In the case of stochastic signal $x(t)$ the convergence of this integral should be understood in the mean square sense [28]. As known, the Hilbert transform performs a $-\pi/2$ phase shift for each frequency component of an arbitrary signal. The instantaneous amplitude $a(t)$ and phase $\Phi(t)$ of $x(t)$ are unambiguously defined through this concept as

$$\Phi(t)=\arctan\left[\frac{y(t)}{x(t)}\right], \quad a^2(t)=x^2(t)+y^2(t), \tag{8}$$

as well as the instantaneous frequency $\omega(t)=\dot{\Phi}(t)$,

$$\omega(t)=\frac{1}{a^2(t)}[x(t)\dot{y}(t)-y(t)\dot{x}(t)]. \tag{9}$$

Afterwards the mean frequency $\langle\omega\rangle$ is given by

$$\langle\omega\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T\omega(t)dt. \tag{10}$$

It is also convenient to introduce the phase difference between the output and input as

$$\phi(t)=\Phi(t)-\Omega_0 t. \tag{11}$$

The concept of the analytical signal can directly be applied to the bistable dynamics (4) in order to derive explicit SDEs for the instantaneous amplitude and the phase difference. In doing so we use the remarkable property of any analytic signal that its Fourier transform vanishes for negative frequencies. Then the SDE for the analytic signal of the periodically driven bistable system reads

$$\dot{w}=w-\frac{1}{4}(3a^2w+w^3)+\Xi(t)+Ae^{i\Omega_0 t}. \tag{12}$$

The analytic noise $\Xi(t)=\xi(t)+i\eta(t)$ with $\eta(t)$ being the Hilbert transformation of $\xi(t)$.

From Eq. (12) we derive the SDEs for the instantaneous amplitude and phase:

$$\dot{a}=a-\frac{1}{2}a^3[1+\cos^2(\phi+\Omega_0 t)]+A\cos\phi+\xi_1(t),$$

$$\dot{\phi}=-\Omega_0-\frac{1}{4}a^2\sin[2(\phi+\Omega_0 t)]-\frac{A}{a}\sin\phi+\frac{1}{a}\xi_2(t), \tag{13}$$

where the noise sources $\xi_{1,2}(t)$ are defined as

$$\xi_1(t)=\xi(t)\cos\Phi+\eta(t)\sin\Phi,$$

$$\xi_2(t)=\eta(t)\cos\Phi-\xi(t)\sin\Phi.$$

Note that Eqs. (13) are exact and similar to those for amplitude and phase fluctuations of a van der Pol oscillator [18]. This similarity arises due to the structure of the nonlinear transformation which we have used. However, there is also an important difference. This distinction appears in the second equation for the phase. In the case of a van der Pol oscillator, an additional term (Ω) occurs in the right-hand side of Eq. (1) remaining in the frequency mismatch. It refers to the natural frequency of the oscillator. The absence of this item in Eq. (13) reflects simply the fact that the overdamped oscillator has no deterministic natural frequency, e.g., there is no rotational term in the equation for the phase of the unperturbed system $(A=0)$.

The exact SDEs (13) are highly nonlinear with multiplicative noise. For computational reasons it is more convenient to integrate original SDE (4) numerically and then to perform the HT by well established techniques (see, for example, [30]). In Fig. 1 we show a typical time series of the state variable $x(t)$, of the instantaneous amplitude $a(t)$, and of the phase $\Phi(t)$.

Clearly, other definitions of the phase are possible too. In particular, for stochastic bistable systems we can introduce an instantaneous phase basing on the occurrence of switchings. Consider interwell switchings: by an appropriate triggering of the original process $x(t)$ the continuous process $x(t)$ can be mapped into a stochastic point process $\{t_k\}$, where t_k are the times of a successive level crossing $x=\pm 1$ (see [10] for details). The residence time between two subsequent switching events is then $T(t)=t_{k+1}-t_k$, $t_k<t<t_{k+1}$. A corresponding dichotomic process $u(t)$ can be introduced via the ansatz

$$u(t)=x_m\mathrm{sgn}[\cos\Phi(t)], \tag{14}$$

where the phase $\Phi(t)$ is defined as [31]

$$\Phi(t)=\pi\frac{t-t_k}{t_{k+1}-t_k}+\pi k, \quad t_k<t<t_{k+1}. \tag{15}$$

A phase defined in this way is a piecewise-linear function of time. In the case of a purely periodic switching process, when transitions between metastable states are fully synchronized with the period $2\pi/\Omega_0$, this definition gives exactly $\Omega_0 t$. The instantaneous frequency $\omega(t)=\pi/T(t)$ is constant during the waiting period $t_k<t<t_{k+1}$ inside a potential well,

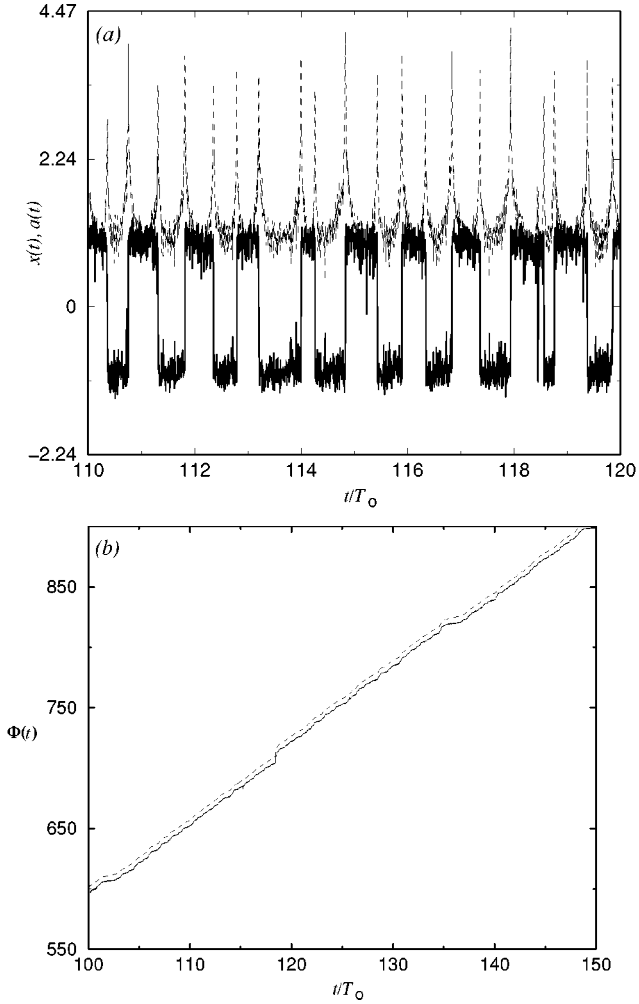

FIG. 1. (a) Time series of the state variable $x(t)$ (solid line) and of the instantaneous amplitude $a(t)$ (dashed line). (b) Time series for instantaneous phase $\Phi(t)$ of the bistable system defined according to the concept of the analytical signal (solid line) and according to the interwell switching analysis (dashed line). Its convergence is nearly perfect. As seen the instantaneous phase of the bistable system except some rare events strictly follow the applied periodic force (linear slope). Phase jumps are accompanied by positive excursions of the instantaneous amplitude. The time axis is given in units of driving period $T_0=2\pi/\Omega_0$. $A=0.089$, $D=0.04$, $\Omega_0=0.002$.

while the mean frequency for this definition is equivalent to the mean switching frequency of the system:

$$\langle\omega\rangle=\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}\frac{\pi}{t_{k+1}-t_k}. \tag{16}$$

It can also be calculated via residence time distribution.

Note that the first definition of the phase bears both interwell and intrawell motions, while the second one takes into account only global switching dynamics. We underline that the first definition does not require the introduction of a threshold value. Nevertheless, both definitions display equivalent averaged behavior up to a constant phase shift. (The problem of constant phase lag between input and periodic response in SR systems versus the noise strength has been discussed in detail in [6]). This coincidence is not by chance. The analytic signal concept causes an automatic separation of different time scales [27]. This follows from the property of the Hilbert transformation to freeze slow variables. In our situation we have fast intrawell fluctuations and slow switchings between metastable states. The global dynamics of the system, e.g., transitions between the metastable states, gives the main contribution to the phase dynamics, while the short-time fluctuations inside a potential well are immaterial for global phase dynamics.

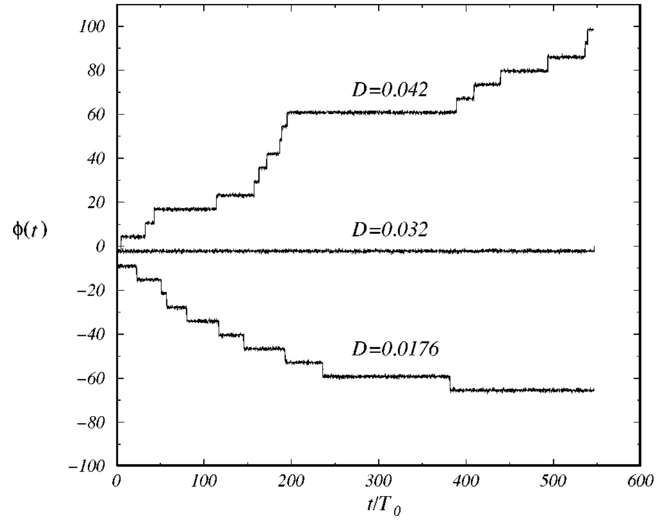

FIG. 2. The instantaneous phase difference calculated using the analytic signal approach for indicated values of noise intensity. For medium noise the phase difference is locked. $A=0.268$, $\Omega_0=0.002$.

IV. NOISE-ENHANCED PHASE LOCKING

Typical time series of the phase difference $\phi(t)=\Phi(t)-\Omega_0 t$ using the analytic signal representation are shown in Fig. 2 for different values of the noise intensity. For small and large noise the switching process and the periodic force are incoherent: on average the phase difference monotonically decreases or increases with time. For weak noise the mean switching frequency is much smaller than the driving frequency and the signal phase surpasses the phase of switchings. On the contrary, for a large noise intensity the signal phase lags behind as the mean switching frequency becomes higher than the driving frequency. However, within some region of noise intensities the phase coherence becomes amenable to observation. This situation is shown in Fig. 2. At an optimal noise level the phase is locked during the course of observation time. As noise intensity deviates from this optimal value the phase slips appear, so that we can speak about partially synchronized dynamics. It is remarkable that the dynamics of the phase difference $\phi(t)$ is very similar to that of a synchronized self-sustained oscillator and can be qualitatively described by the SDE (1) with coefficients depending on the noise intensity, driving amplitude, and frequency.

Figure 2 clearly shows the effect of synchronization: the phases of the switching process and of the input signal are *instantaneously* locked at an optimal noise level. It is also seen from this figure that by tuning noise we can increase the duration of time intervals of locking. We remark that the

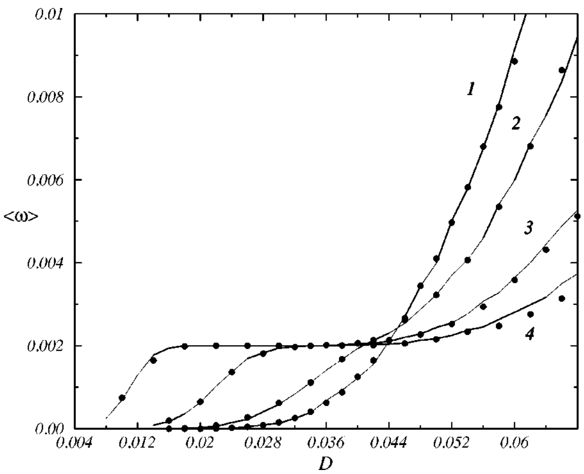

FIG. 3. Mean frequency (10) (solid line) and the mean switching frequency (16) (symbols) versus noise intensity for different values of driving amplitude. For sufficiently large amplitudes over a finite range of noise intensity the frequencies are locked to the frequency of the driving signal. $A=0$ (1), $A=0.089$ (2), $A=0.178$ (3), and $A=0.268$ (4); $\Omega_0=0.002$.

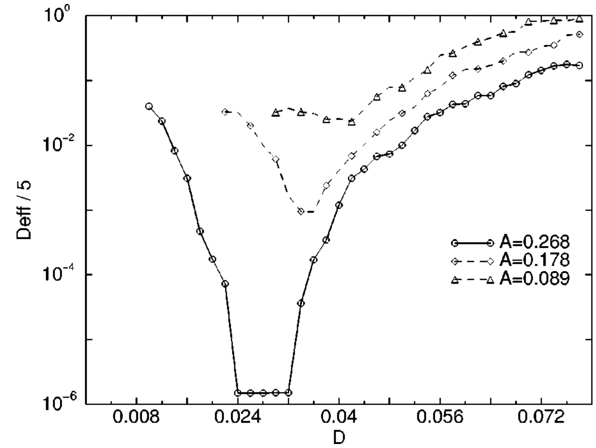

FIG. 4. The effective diffusion constant versus noise intensity for indicated values of driving amplitude for $\Omega_0=0.002$.

same behavior has also been observed for phases determined via switching times (15).

The mean frequency, determined via the analytic signal concept (10), and of the mean switching frequency, calculated by averaging the residence times (16) is shown in Fig. 3 versus noise intensity for different values of the driving amplitude. Again this figure displays the effect of locking the mean switching frequency reported in [13]. For vanishingly small signals the mean frequency follows the Kramers law and grows exponentially with increasing noise strength. However, with a sufficiently large A the mean frequency matches the driving frequency in a finite region of the noise intensity. Note that the behavior of the mean frequencies calculated using two different definitions of instantaneous phase is nearly converging. It is important to mention that the effect of mean frequency locking occurs in a finite region of noise intensities. The width of this region depends on the driving amplitude and the frequency [13].

Although the effects of the phase and of the mean frequency locking already indicate a synchronizationlike behavior we need to calculate second-order statistical quantities in order to determine synchronization according to the definitions given above. We aim to answer the question: how long is the phase at the output locked by the signal? For this purpose we calculate the effective diffusion constant.

The dependence of the effective diffusion constant (2) vs noise intensity is shown in Fig. 4 for different values of driving amplitude. In contrast to classical oscillators, where D_{eff} monotonically increases, here the effective diffusion constant passes through a minimum. This means in its turn that the phase becomes locked for longer time intervals with the increase of the noise intensity. In other words, we can enhance phase coherence by increasing the noise level in the system. This can be considered as a new manifestation of stochastic resonance.

It is important to underline that effects of phase and frequency locking occur for strong (but undercritical) signals only. For a weak signal the system is only partially synchronized even in the case when the mean switching frequency equals exactly the driving frequency. This situation is shown in Fig. 5. Although there are relatively short locking segments, the phase difference displays random-walk-like behavior without a preferred slope.

From Fig. 4 we conclude that for sufficiently strong signals the diffusion of the phase difference is extremely small in a finite region of noise intensities. It enables us to define regions of effective synchronization in the parameter space. We indicate a system (4) as effectively synchronized to an external periodic signal if its instantaneous phase is locked during 100 periods of the signal. This condition is expressed as $D_{\mathrm{eff}}\leqslant 2\pi\Omega_0/100$.

Regions of synchronization in the parameter plane A-D are shown in Fig. 6 for different values of driving frequency. A periodic force with amplitude less than the presented curves does not synchronize the bistable system in the above defined sense. The regions have a tonguelike shape. The thresholdlike character of the synchronization is clearly seen. It means that for a given frequency the minimal amplitude necessary for synchronization never vanishes. Recall that experimentally obtained ''Arnold tongues'' of a periodically driven noisy Schmitt trigger [13] also have the same thresh-

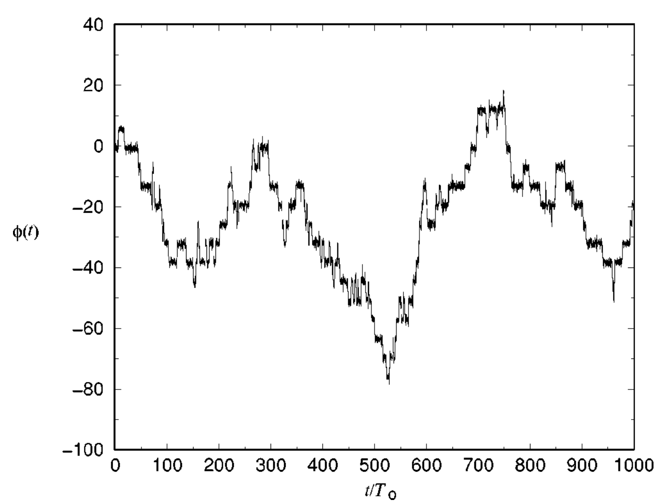

FIG. 5. The phase difference versus time (in units of driving period) for small amplitudes. Only partial synchronization over short-time segments can be observed. $A=0.089$, $D=0.04$.

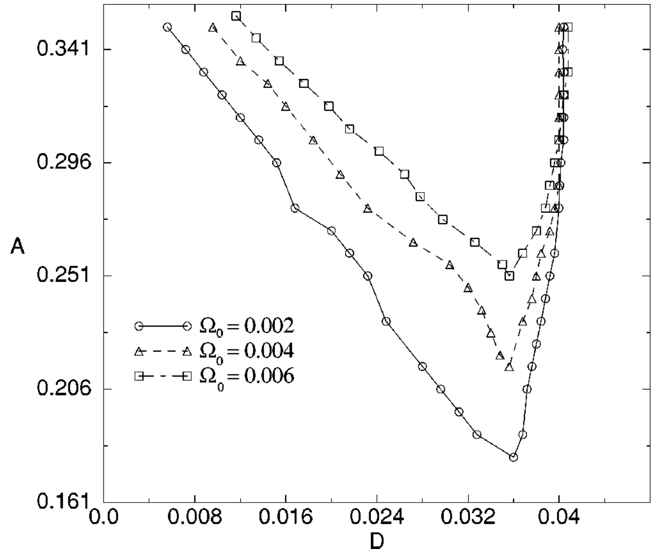

FIG. 6. The effective synchronization regions for indicated values of driving frequency. The output is effectively synchronized for amplitudes above the presented curves. To achieve synchronized states for a given frequency there exists a nonvanishing threshold for the amplitudes.

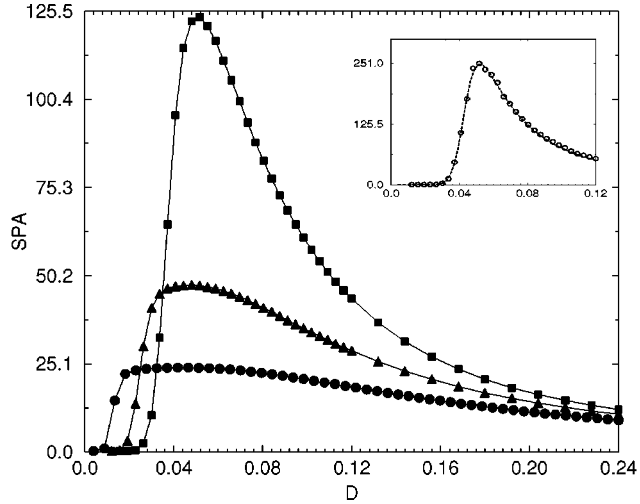

FIG. 7. SPA versus noise intensity for different values of the driving amplitude: $A=0.089$ (boxes), $A=0.179$ (triangles), $A=0.268$ (circles). Inset: SPA versus D for $A=0.02$. The linear response approximation is shown by the dashed line. $\Omega_0=0.002$.

old feature. With the increase of the driving frequency the threshold value of the driving amplitude also increases and the region of synchronization shrinks. This feature is determined by the low-frequency character of SR in bistable systems. Our numerical experiments have verified that the measures of phase coherence do not display any resonancelike (nonmonotonous) behavior as functions of driving frequency.

Qualitatively the same results have been obtained for another representative of SR systems, the Fitz Hugh-Nagumo neuron model [32]. SR and synchronization in this model have been extensively studied in [33,15]. The basic features in this case can be gained by studying the spike trains generated by the system. A suitable definition of the instantaneous phase is given by

$$\Phi(t)=2\pi\frac{t-\tau_k}{\tau_{k+1}-\tau_k}+2\pi k, \quad t_k<t<t_{k+1} \tag{17}$$

where the τ_k label times of firing events.

V. SPECTRAL POWER AMPLIFICATION, RESIDENCE TIME DISTRIBUTIONS, AND PHASE SYNCHRONIZATION

In the traditional description of SR in terms of spectral density, information about the instantaneous phase is lost. In Fig. 7 we show numerically obtained SPA versus noise intensity for different value of the driving amplitude. In the inset we show the SPA for an extremely weak signal and the approximation according to linear response theory [4]. The SPA decreases with the increase of the driving amplitude and the value of optimal noise intensity at which the SPA achieves maximal values shifts to lower noise level [4]. At the same time, the shape of the SPA curve flattens as the amplitude of the signal increases. However, the effect of SR still exists. The SPA takes its maximum at a certain value of noise intensity.

It is important to note that the values of noise intensity maximizing the SPA approximately correspond to those at which the effective diffusion constant is minimal, i.e., to the most phase coherent state of the system. On the other hand, the signal-to-noise ratio is maximized at sufficiently larger values of noise intensity.

The residence time distribution gives a weaker definition of synchronization in SR systems based on the restriction imposed to the frequency fluctuations. This approach defines synchronization in an averaged sense. Really, the existence of the peak at the half driving period indicates that the number of residence times which are near the half driving period is much larger than the whole number of switchings occurring during an observation time. However, it does not require instantaneous phase locking during the course of long times. Therefore the residence time distribution recovers an average phase preference of the system. That is why the measures based on the residence time distribution reflect the synchronization nature of SR even for weak signals.

Let us take a comparatively small amplitude of the signal, $A=0.089$. The residence time distributions are shown in Fig. 8 for different noise intensities. Although the residence time distribution reflects synchronizationlike behavior, the phase dynamics cannot be viewed as synchronized: even when the mean switching frequency equals the driving frequency, $\langle\omega\rangle=\Omega_0$, the phase difference performs Brownianlike motion with zero slope (see Fig. 5). In the synchronization (phase locking) region the residence time distribution is represented by a single narrow peak at the half driving period.

A meaningful criterion of SR based on the residence time distribution that has been proposed recently in [12] is the height of the peak of the residence time distribution minus unmodulated residence time distribution (the deviation from the unperturbed residence time distribution) at the half driving period. This quantity, labeled as *a*, is shown in Fig. 9 for the different values of driving amplitude as a function of noise intensity. With the increase of driving amplitude, *a*

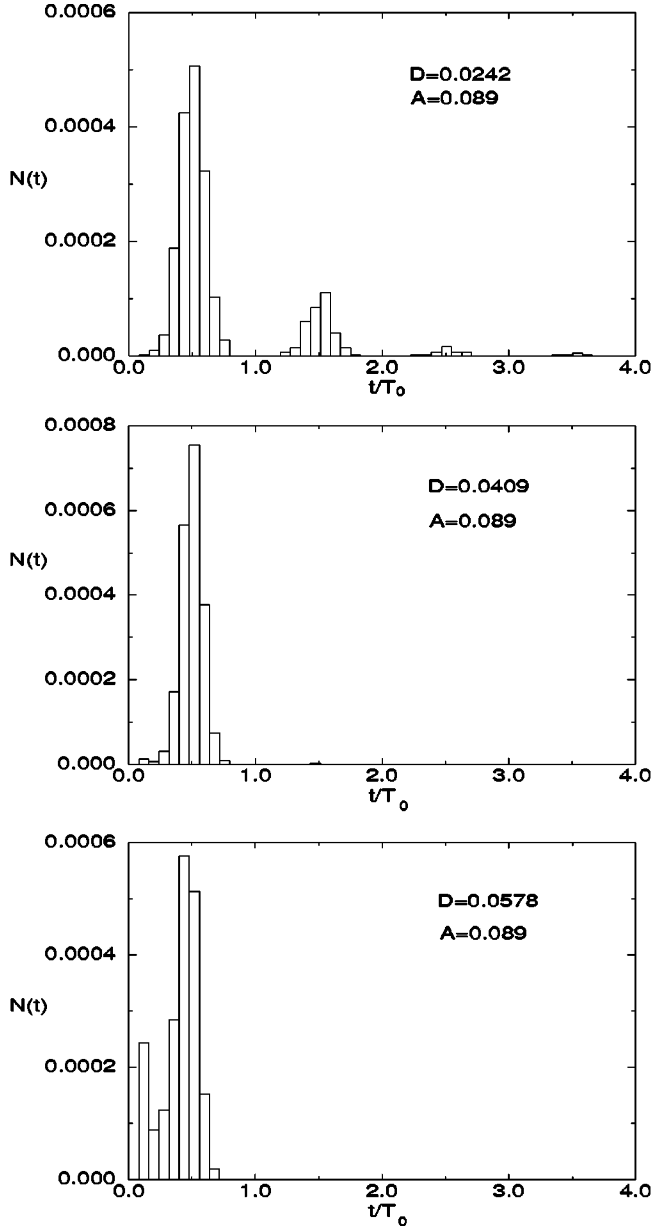

FIG. 8. Residence time distributions for indicated values of noise intensity and driving amplitude. $\Omega = 0.002$.

also increases and the value of noise intensity which maximizes a is shifted towards smaller values of D.

VI. CONCLUSION

We have studied SR in classical terms of phase synchronization. We used two definitions of the instantaneous phase,

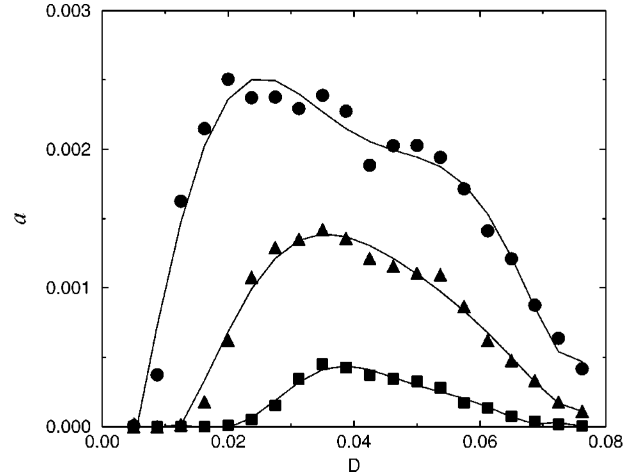

FIG. 9. Height of the first peak in the residence time distribution with extracted unmodulated residence time distribution vs noise intensity for different values of the driving amplitude: $A = 0.089$ (boxes), $A = 0.179$ (triangles), $A = 0.268$ (circles). $\Omega = 0.002$.

basing on the analytic signal concept and on the switching time sequences. Both phase definitions provide the same results for averaged quantities. The effect of phase synchronization of the stochastic switching process is shown to occur in finite regions of the noise intensities. However, this effect is restricted by comparatively large amplitudes of external signal. A measure of phase coherence, the effective diffusion constant, passes through a minimum being plotted versus noise intensity. Therefore stochastic resonance manifests itself as a phenomenon of noise-enhanced phase coherence. This noise-enhanced ordering is also reflected in the nonmonotonous behavior of the source entropy as was found in [34].

For a weaker signal the synchronization features of SR systems can be gained by the residence time distribution. This approach gives an averaged description of SR as a synchronization phenomenon. However, synchronization is absent for extremely weak signals.

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